A NOTE ON COMMUTING DIFFEOMORPHISMS ON SURFACES

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Abstract. Let Σ be a closed surface with nonzero Euler characteristic. We prove the existence of an open neighborhood $\mathcal{V}$ of the identity map of Σ in the C^1-topology with the following property: if G is an abelian subgroup of $\mathrm{Diff}^1(\Sigma)$ generated by any family of elements in $\mathcal{V}$ then the elements of G have common fixed points. This result generalizes a similar result due to Bonatti and announced in his paper *Difféomorphismes commutants des surfaces et stabilité des fibrations en tores*.

1. Introduction

Bonatti has proven in [2] the following result.

Bonatti's Theorem. *Let Σ be a closed surface with nonzero Euler characteristic. Fixed $k \in \mathbb{Z}^+$, there is an open C^1-neighborhood $\mathcal{U}_k$ of the identity map of Σ satisfying the following: if G is an abelian subgroup of $\mathrm{Diff}^1(\Sigma)$ generated by k elements in $\mathcal{U}_k$ then for some $p \in \Sigma$ we have $f(p) = p$ for all $f \in G$.*

We remark that in the above theorem the size of the neighborhood $\mathcal{U}_k$ depends on the number k of generators of the abelian group G unless Σ is the 2-sphere S^2 or the projective plane $\mathbb{RP}^2$. In fact, the cases S^2 and $\mathbb{RP}^2$ were treated by Bonatti in [1] and in these cases the neighborhood of the identity map can be chosen to be uniform.

The purpose of this paper is to prove that even when Σ is different from S^2 and $\mathbb{RP}^2$, there exists a distinguished C^1-neighborhood of the identity map of Σ where the above theorem holds regardless of the number of generators of the group G. Precisely, we prove the following theorem.

Theorem 1.1. *Let Σ be a closed surface with nonzero Euler characteristic. Then there exists an open C^1-neighborhood $\mathcal{V}$ of the identity map of Σ having the following property: if G is an abelian subgroup of $\mathrm{Diff}^1(\Sigma)$ generated by any family of elements in $\mathcal{V}$ then for some $p \in \Sigma$ we have $f(p) = p$ for all $f \in G$.*

With regard to the techniques of Bonatti's paper [2] we observe that he provides a neighborhood $\mathcal{U}_2$ of the identity map of Σ such that two commuting diffeomorphisms in this neighborhood have common fixed points. To guarantee the existence of a common fixed point for three commuting diffeomorphisms, he needs to shrink the neighborhood $\mathcal{U}_2$ to a certain neighborhood $\mathcal{U}_3$. Similarly, $\mathcal{U}_3$ have to be shrink if there is four or more diffeomorphisms. Our argument consists of showing that the above mentioned sequence $\mathcal{U}_2, \mathcal{U}_3, \ldots$ of neighborhoods actually stabilizes at some integer depending only on the topology of the surface.

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Since we are dealing with abelian groups generated by diffeomorphisms close to the identity map their lifts to a double covering space still form an abelian group with generators close to the identity. Therefore, by using the double covering of the orientations of Σ we conclude that to prove Theorem 1.1 it suffices to prove it for orientable closed surfaces with nonzero Euler characteristic.

We close the introduction with the following question.

Does our theorem hold for homeomorphisms C^0 close to the identity map of a closed surface Σ? It seems to the author that this question remains open even for C^1-diffeomorphisms that are C^0 close to the identity map. Handel, in [6], proves the existence of common fixed points for two commuting homeomorphisms of S^2 which are, for example, C^0 close to the identity map. Moreover, as to higher genus surfaces, he proves that two orientation preserving C^1-diffeomorphisms f and g which commute have at least as many common fixed points as F and G do, provided that F and G are pseudo-Anosov. Here, F and G stand for the homeomorphisms respectively obtained from f and g by the Thurston Classification Theorem for surface homeomorphisms.

2. Notations and definitions

From now on Σ will be a closed connected oriented surface embedded in the Euclidean space $\mathbb{R}^3$ endowed with the usual norm denoted by $\|\cdot\|$. The distance in Σ associated to the induced Riemannian metric from $\mathbb{R}^3$ will be denoted by d and we denote by $\mathrm{B}(x\,;\rho)$ the closed 2-ball centered at x with radius $\rho \in (0\,,\infty)$ with respect to d.

The C^1-norm of a given C^1-map $\varphi : \Sigma \to \mathbb{R}^3$ is defined by

$$\|\varphi\|_1 = \sup_{x\in\Sigma}\Big\{\|\varphi(x)\| + \sup_{v\in T_x\Sigma\,;\,\|v\|=1}\|D\varphi(x)\cdot v\|\Big\}.$$

Defined on the real vector space consisting of C^1-maps from Σ to $\mathbb{R}^3$, this norm induces a distance between C^1-maps $\varphi, \psi : \Sigma \to \mathbb{R}^3$ given by $\|\varphi - \psi\|_1$. The distance between two C^1-diffeomorphisms f, h of Σ is, by definition, the distance between f and h as C^1-maps from Σ to $\mathbb{R}^3$. This distance on the space of all C^1-diffeomorphisms of Σ defines the so-called C^1-topology and the group of C^1-diffeomorphisms of Σ endowed with this topology will be denoted by $\mathrm{Diff}^1(\Sigma)$.

Given a list $\mathcal{H}$ of elements in $\mathrm{Diff}^1(\Sigma)$, let $\mathrm{Fix}(\mathcal{H})$ denote the set of common fixed points of the elements of $\mathcal{H}$. In other words, $\mathrm{Fix}(\mathcal{H}) = \bigcap_{f\in\mathcal{H}}\mathrm{Fix}(f)$ where $\mathrm{Fix}(f)$ is the set of fixed points of f.

The *positive semi-orbit* of $p \in \Sigma$ by a diffeomorphism f is the set $\mathcal{O}^+_p(f) = \{f^n(p)\ ;\ n \geq 0\}$. Its closure in Σ will be denoted by $\overline{\mathcal{O}^+_p(f)}$. We say that $p \in \Sigma$ is a *ω-recurrent* point for f if p is the limit of some subsequence of $\big(f^n(p)\big)_{n\geq 0}$.

Let $a, b \in \Sigma$ be such that $d(a,b)$ is smaller than the injectivity radius of the exponential map associated to the metric of Σ. For such $a\,,b$ we denote by $[\,a\,,b\,]$ the oriented geodesic arc joining a to b which is contained in the disc of injectivity centered at a.

Fix $\kappa > 0$ such that $d(a,b)$ is smaller than the injectivity radius of the exponential map whenever $\|b - a\| < \kappa$.

Now, given $f \in \mathrm{Diff}^1(\Sigma)$ such that $\|f - \mathrm{Id}\|_1 < \kappa$, let X_f be the standard vector field on Σ associated to f as follows: $X_f(x)$ is the tangent vector to the geodesic segment $[\,x\,,f(x)\,]$ at the point x with orientation given by the orientation

of $[\,x\,,f(x)\,]$ whose norm is equal to $\|f(x)-x\|$. The singular set of X_f is the set $\mathrm{Fix}(f)$.

Moreover, let $p\in\Sigma-\mathrm{Fix}(f)$. Following Bonatti [2], a piecewise geodesic simple closed curve Γ_f^p on Σ is said to be *supported by* $\mathcal{O}_p^+(f)$ if each geodesic arc of Γ_f^p is contained in some segment $[f^i(p)\,,f^j(p)]$ satisfying

$$d\big(f^i(p)\,,f^j(p)\big)\leq \tfrac{3}{2}\,d\big(f^i(p)\,,f^{i+1}(p)\big)\quad\text{where}\quad i\,,j\geq 0.$$

Consider $0<\alpha<\pi$. We say that Γ_f^p is *α-tangent* to the vector field Y provided that the following conditions are satisfied: Y has no singularities along Γ_f^p and for one of the two possible orientations of Γ_f^p and for each point x of a geodesic arc Ω of Γ_f^p the angle between the vector $Y(x)$ and the unitary tangent vector to Ω at x induced by the orientation of Γ_f^p is less than α for all $x\in\Omega$.

Furthermore, let γ_f^p denote the curve obtained by concatenating the segments $[f^i(p)\,,f^{i+1}(p)]$ for $i\geq 0$.

The following topological result about compact surfaces will be very important in our proofs. It will be used in the proof of Theorem 5.1 and in the proof of the Main Lemma.

Let $\Sigma\subset\mathbb{R}^3$ be an oriented connected closed surface. Then there is an integer $N\geq 4$ with the following property:

Given any compact connected surface $\mathcal{S}\subset\Sigma$ with boundary such that each connected component of its boundary is not null homotopic in Σ, one has:

- The number of connected components of $\partial\mathcal{S}$ is less than N;
- If $\alpha_1,\dots,\alpha_N\subset\mathrm{Int}(\mathcal{S})$ is a list of pairwise disjoint simple closed curves then there are two distinct curves $\alpha_i\,,\alpha_j$ in that list which are homotopic in $\mathcal{S}$. Consequently,
  - either each one of these two curves bounds disks embedded in $\mathcal{S}$;
  - or these two curves bound a cylinder embedded in $\mathcal{S}$.

We remark that the constant $N\geq 4$ considered in the last paragraph will frequently be used in all this paper.

3. Some known results

In this section we recall some technical results from [2] which play a key role in this note.

Given $\epsilon>0$ let $\{\mathcal{V}_k(\epsilon)\}_{k\geq 1}$ be a decreasing nested sequence of open neighborhoods of the identity map of Σ in the C^1-topology inductively defined as follows:

- $\mathcal{V}_1(\epsilon)=\big\{f\in\mathrm{Diff}^1(\Sigma)\ ;\ \|f-\mathrm{Id}\|_1<\epsilon\big\}$;
- Fixed $\mathcal{V}_k(\epsilon)$ for some positive integer k we choose $\mathcal{V}_{k+1}(\epsilon)$ so that the following holds:

  $$\text{if}\quad f_1,\dots,f_{2N+1}\in\mathcal{V}_{k+1}(\epsilon)\quad\text{then}\quad f_1\circ f_2\circ\cdots\circ f_{2N}\circ f_{2N+1}\in\mathcal{V}_k(\epsilon)$$

  where $f_1\circ f_2\circ\cdots\circ f_{2N}\circ f_{2N+1}$ stands for the composition of maps.

We always assume that $\epsilon<\kappa$ so that we can guarantee that $[\,p\,,f(p)]$ is well defined whenever $f\in\mathcal{V}_1(\epsilon)$.

Lemma 3.1. (Bonatti) *There exists $0<\epsilon_1<\kappa$ such that every pair of elements $f,h\in\mathrm{Diff}^1(\Sigma)$ satisfies the following:*

(1) *If* $f \in \mathcal{V}_1(\epsilon_1)$ *and* $f(p) \neq p$ *then* f *does not have fixed points in the ball* $\mathrm{B}\big(p\,;4\,d(p\,,f(p))\big)$*. In particular,* f *does not have fixed points along curves supported by* $\mathcal{O}_p^+(f)$ *if such curves exist.*
(2) *If* $f \in \mathcal{V}_1(\epsilon_1)$ *and* $p \in \Sigma - \mathrm{Fix}(f)$ *is a* ω*-recurrent point of* f *then there is a curve* Γ_f^p *supported by* $\mathcal{O}_p^+(f)$ *and* $\frac{\pi}{10}$*-tangent to* X_f*.*
(3) *If* $f,h \in \mathcal{V}_2(\epsilon_1)$ *commute and* $p_i \in \mathrm{Fix}(h\circ f^i) - \mathrm{Fix}(f)$ *then* $h\circ f^j$ *has no fixed points in the ball* $\mathrm{B}\big(p_i\,;4\,d(p_i\,,f(p_i)\big)$ *where* $j \neq i$ *and* $i\,,j \in \{0,\ldots,2N\}$*. In particular,* $h\circ f^j$ *does not have fixed points along curves supported by* $\mathcal{O}_{p_i}^+(f)$*.*
(4) *If* $f,h \in \mathcal{V}_2(\epsilon_1)$ *commute,* $0 < d(p_i\,,p) \le \frac{3}{2}\,d(p_i\,,f(p_i))$ *and the segment* $[\,p_i\,,p\,]$ *is* $\frac{\pi}{10}$*-tangent to the vector field* X_f *then the segment* $[\,p_i\,,p\,]$ *is* $\frac{2\pi}{5}$*-tangent to the vector field* $X_{h\,\circ\,f^j}$ *for all* $j \neq i$*. Here* $i\,,j \in \{0,1,\ldots,2N\}$*,* $p \in \Sigma$ *and* $p_i \in \mathrm{Fix}(h\circ f^i) - \mathrm{Fix}(f)$*.*

The simple closed curve Γ_f^p obtained in item (2) of Lemma 3.1 will be called *character curve* of f at p.

The results listed in Lemma 3.1 are proven in [2] for the integers $i\,,j \in \{0,\ldots,N\}$ where N was defined at the end of the last section. Nonetheless, it is easy to see that the proofs in [2] also work for an arbitrary positive integer modulo reducing ϵ_1.

4. Preparing the proof of Theorem 1.1

In this section we shall establish three lemmas which will be used in the next section to prove Theorem 1.1. The proof of the first lemma is, in fact, contained in the proof of Lemma 4.1 of [1, pages 67–68]. We repeat the arguments here because our hypothesis are not exactly the same as those used in Bonatti's paper. Besides, in section 8, the argument below will be further adapted to apply to more general and technical situations. Lemma 4.1 below says that curves supported by positive semi-orbits are disjoint under appropriate conditions.

From now on the neighborhood $\mathcal{V}_k(\epsilon_1)$ will be denoted simply by $\mathcal{V}_k$ for all $k \in \mathbb{Z}^+$ where ϵ_1 is always given by Lemma 3.1.

Lemma 4.1. *Let* $f,h \in \mathcal{V}_1$ *be commuting diffeomorphisms and let* Γ_f^p,Γ_h^q *be curves supported respectively by* $\mathcal{O}_p^+(f)\,,\mathcal{O}_q^+(h)$ *where*

$$p \in \mathrm{Fix}(h) - \mathrm{Fix}(f) \quad \textit{and} \quad q \in \mathrm{Fix}(f) - \mathrm{Fix}(h).$$

Then we have:

- $\gamma_f^p \cap \Gamma_h^q = \emptyset\,;$
- $\Gamma_f^p \cap \Gamma_h^q = \emptyset\,;$
- $\overline{\mathcal{O}_p^+(f)} \cap \Gamma_h^q = \emptyset.$

Moreover, if $\rho > 0$ *is such that*

$$d\big(x\,,f(x)\big)\,,d\big(y\,,h(y)\big) \ge \rho \quad , \quad \forall\, x \in \mathcal{O}_p^+(f) \quad \textit{and} \quad \forall\, y \in \mathcal{O}_q^+(h)$$

then $d(\Gamma_f^p,\Gamma_h^q) \ge \rho$*.*

Proof. Let us first prove the second item. To do this we suppose for a contradiction that $\Gamma_f^p \cap \Gamma_h^q \neq \emptyset$. Thus, there exist integers $m,n,k,l \ge 0$ such that

$$[f^m(p)\,,f^n(p)] \cap [h^k(q)\,,h^l(q)] \neq \emptyset$$

where

$$\begin{aligned} d\big(f^m(p)\,,f^n(p)\big) &\le \tfrac{3}{2}\,d\big(f^m(p)\,,f^{m+1}(p)\big)\\ d\big(h^k(q)\,,h^l(q)\big) &\le \tfrac{3}{2}\,d\big(h^k(q)\,,h^{k+1}(q)\big). \end{aligned} \tag{4.1.1}$$

By the triangle inequality we obtain:

$$\begin{aligned} d\big(f^m(p)\,,h^k(q)\big) &\le d\big(f^m(p)\,,f^n(p)\big) + d\big(h^k(q)\,,h^l(q)\big)\\ &\le \tfrac{3}{2}\,d\big(f^m(p)\,,f^{m+1}(p)\big) + \tfrac{3}{2}\,d\big(h^k(q)\,,h^{k+1}(q)\big)\\ &\le 3\max\Big\{d\big(f^m(p)\,,f^{m+1}(p)\big)\,,\ d\big(h^k(q)\,,h^{k+1}(q)\big)\Big\}. \end{aligned}$$

Therefore, we have the following two possibilities:

- either $h^k(q)$ is in the ball $\mathrm{B}\big(f^m(p)\,;3\,d(f^m(p)\,,f^{m+1}(p))\big)$ which is impossible by item (1) of Lemma 3.1 since the map f has no fixed points in $\mathrm{B}\big(f^m(p)\,;3\,d(f^m(p)\,,f^{m+1}(p))\big)$. Note that $h^k(q)\in\mathrm{Fix}(f)$ which follows from the commutativity;
- or $f^m(p)$ is in the ball $\mathrm{B}\big(h^k(q)\,;3\,d(h^k(q)\,,h^{k+1}(q))\big)$ which is impossible by the same reason.

This finish the proof of the second item.

The reader will notice that the above arguments prove also the first item. The last item follows from observing that $\overline{\mathcal{O}_p^+(f)}\subset\mathrm{Fix}(h)$ and h is free of fixed points over Γ_h^q thanks to item (1) of Lemma 3.1.

To prove the second part of the lemma let us suppose for a contradiction that $d(\Gamma_f^p,\Gamma_h^q)<\rho$. Then there exist $m,n,k,l\ge 0$ satisfying (4.1.1) and two points

$$a\in[f^m(p)\,,f^n(p)]\quad\text{and}\quad b\in[h^k(q)\,,h^l(q)]$$

such that $d(a\,,b)<\rho$. Therefore,

$$\begin{aligned} d\big(f^m(p)\,,h^k(q)\big) &\le d\big(f^m(p)\,,f^n(p)\big) + d(a\,,b) + d\big(h^k(q)\,,h^l(q)\big)\\ &\le \tfrac{3}{2}\,d\big(f^m(p)\,,f^{m+1}(p)\big) + \rho + \tfrac{3}{2}\,d\big(h^k(q)\,,h^{k+1}(q)\big)\\ &\le 4\max\Big\{d\big(f^m(p)\,,f^{m+1}(p)\big)\,,\ d\big(h^k(q)\,,h^{k+1}(q)\big)\Big\}. \end{aligned}$$

Now, we finish the proof by using exactly the same arguments used to prove that $\Gamma_f^p\cap\Gamma_h^q=\emptyset$. □

The next lemma is a version of Lemma 5.1 in [1, page 69] for the surface Σ. It is a version of Bonatti's Theorem in [2] for a special kind of boundary.

Here we use the notation $x\in\mathrm{Fix}(f_1,\dots,\widehat{f_\lambda},\dots,f_m)$ to mean that $x\in\mathrm{Fix}(f_i)$ for all $i\in\{1,\dots,m\}$ and $i\ne\lambda$.

Lemma 4.2. *Let $f_1,\dots,f_{3N}\in\mathcal{V}_{3N+1}$ be commuting diffeomorphisms and let $\mathcal{S}\subset\Sigma$ be a compact connected surface with $\chi(\mathcal{S})\ne 0$. Suppose that any connected component Γ of $\partial\mathcal{S}$ satisfies the following:*

- *Γ is not null homotopic in Σ;*
- *Γ is a character curve of f_λ at*

$$p\in\mathrm{Fix}(f_1\,,\dots,\widehat{f_\lambda}\,,\dots,f_{3N})-\mathrm{Fix}(f_\lambda)$$

for some $\lambda\in\{1,\dots,3N\}$.

Then $f_1,\dots,f_{3N}$ have common fixed points in $\mathrm{Int}(\mathcal{S})$.

We notice that this lemma contains the case $\mathcal{S}=\Sigma$. Moreover, by the definition of the integer N we have that the number of connected components of $\partial\mathcal{S}$ is less than N. The proof of this lemma will be deferred to section 7.

The proof of the next lemma is easily obtained from the proof of Lemma 5.1 in [1, page 69].

Lemma 4.3. *Let $f_1, \dots, f_n \in \mathcal{V}_1$ be commuting diffeomorphisms and let $\Gamma^p_{f_n}$ be a character curve of f_n at*

$$p \in \mathrm{Fix}(f_1, \dots, f_{n-1}) - \mathrm{Fix}(f_n).$$

If $\Gamma^p_{f_n}$ bounds a disc in Σ then $f_1, \dots, f_n$ have a common fixed point in the interior of that disc.

5. Proof of Theorem 1.1

Theorem 1.1 will be obtained as an easy consequence of Theorem 5.1 to be stated and proved below. The proof of Theorem 5.1 is by induction and by contradiction. It consists of two parts. In the first part the induction procedure is initialized (i.e. the first step of the induction is stablished). This part will be carried after the proof of Lemma 4.2 in section 7. There, we shall recast Bonatti's proof [2] in a more general context (with boundaries) for k diffeomorphisms of Σ where $2 \le k \le 3N$ (recall that N was defined at the end of section 3). This argument will also go by induction and by contradiction. At each step of the induction procedure over k, the neighborhood of the identity in question will be reduced to guarantee the existence of a common fixed point for the diffeomorphisms. Roughly speaking, the technical reason to reduce the neighborhood is that we need to construct N special character curves pairwise disjoint by using $k+1$ diffeomorphisms $f_1, \dots, f_k, f_{k+1}$ (once the existence of the common fixed point for k diffeomorphisms has been stablished). The construction of these character curves is carried out by using the positive semi-orbit by f_{k+1} of common fixed points of the k diffeomorphisms $f_1, \dots, f_{k-1}, f_k \circ f^i_{k+1}$ where $i \in \{1, \dots, N\}$. In view of the diffeomorphism $f_k \circ f^i_{k+1}$, the neighborhood of the identity needs to be reduced so as to guarantee that $f_k \circ f^i_{k+1}$ belongs to the neighborhood obtained in the previous step of the induction (i.e. the case of k diffeomorphisms). This technical question is already apparent in Bonatti's proof of [2, page 109].

In the second part of the proof we have a family consisting of more than $3N$ commuting diffeomorphisms. In this case these N special character curves pairwise disjoint will be obtained through the positive semi-orbit by f_j of the common fixed points of the diffeomorphisms $f_1, \dots, \widehat{f_j}, \dots, f_{3N+n+1}$ where f_j will be conveniently chosen from the set $\{f_{N+1}, \dots, f_{3N}\}$. Finally, with this new construction procedure of the N pairwise disjoint character curves, we shall be able to keep the same neighborhood of the identity for $3N$ diffeomorphisms.

The construction of the character curves carried out in the second part of the proof, which exploits the existence of a large number of generators, is the essential difference between Bonatti's proof and the present one. Naturally, in order to apply this strategy, Bonatti's Theorem has to be extended to a more general settings. Such extension however will be accomplished by using the same arguments employed in [1, 2].

On the other hand, to construct these N character curves supported by appropriate semi-orbits, it will be necessary to ensure that the semi-orbits remain in $\mathrm{Int}(\mathcal{S})$. It will also be necessary to guarantee that the corresponding character curves are disjoint and do not intersect the boundary of $\mathcal{S}$. For all that, Lemma 4.1 will be crucial.

As a matter of fact, the strategy used in the second part is implicit in the proof of Bonatti's Theorem for S^2 in [1]. In this case, only two diffeomorphisms are needed to implement the construction of the character curves. As a consequence, the first part of the proof is superfluous in this case.

Theorem 5.1. *Let $n \geq 0$ be an integer and let*

$$f_1, \dots, f_N, h_1, \dots, h_{2N+n} \in \mathcal{V}_{3N+1}$$

be commuting diffeomorphisms. Consider a compact connected surface $\mathcal{S} \subset \Sigma$ with $\chi(\mathcal{S}) \neq 0$. Suppose that any connected component Γ of $\partial\mathcal{S}$ satisfies the following:

- *Γ is not null homotopic in Σ;*
- *Γ is a character curve of f_λ at*

$$p \in \mathrm{Fix}(f_1, \dots, \widehat{f_\lambda}, \dots, f_N, h_1, \dots, h_{2N+n}) - \mathrm{Fix}(f_\lambda)$$

for some $\lambda \in \{1, \dots, N\}$.

Then $f_1, \dots, f_N, h_1, \dots, h_{2N+n}$ have common fixed points in $\mathrm{Int}(\mathcal{S})$.

Proof. We argue by induction on $n \geq 0$.

The theorem holds for $n = 0$ since this case reduces to Lemma 4.2.

Now, let us assume that it holds for some integer $n \geq 0$. Also, let us suppose for a contradiction that $f_1, \dots, f_N, h_1, \dots, h_{2N+n}, h_{2N+n+1}$ do not have common fixed points in $\mathrm{Int}(\mathcal{S})$.

By assumption, if $\partial\mathcal{S} \neq \emptyset$ then each connected component of $\partial\mathcal{S}$ is a character curve $\Gamma_{f_\lambda}^{p_\lambda}$ of f_λ at

$$p_\lambda \in \mathrm{Fix}(f_1, \dots, \widehat{f_\lambda}, \dots, f_N, h_1, \dots, h_{2N+n}, h_{2N+n+1}) - \mathrm{Fix}(f_\lambda)$$

for some $\lambda \in \{1, \dots, N\}$.

For each $j \in \{1, \dots, N\}$ let us consider the list

$$f_1, \dots, f_N, h_1, \dots, \widehat{h_j}, \dots, h_N, \dots, h_{2N+n+1}$$

of $3N + n$ diffeomorphisms. From the induction assumption on n we conclude that there exists a point $q_j \in \mathrm{Int}(\mathcal{S})$ such that

$$q_j \in \mathrm{Fix}(f_1, \dots, f_N, h_1, \dots, \widehat{h_j}, \dots, h_{2N+n+1}) - \mathrm{Fix}(h_j)$$

since $f_1, \dots, f_N, h_1, \dots, h_{2N+n+1}$ do not have common fixed points in $\mathrm{Int}(\mathcal{S})$. From Lemma 4.1 we know that $\overline{\mathcal{O}_{q_j}^+(h_j)} \subset \mathrm{Int}(\mathcal{S})$. On the other hand, $\overline{\mathcal{O}_{q_j}^+(h_j)}$ is invariant by h_j and contained in

$$\mathrm{Fix}(f_1, \dots, f_N, h_1, \dots, \widehat{h_j}, \dots, h_{2N+n+1}) - \mathrm{Fix}(h_j).$$

Now, thanks to Zorn's Lemma, we can assume without loss of generality that q_j is a ω-recurrent point for h_j.

Besides, the maps $f_1, \dots, f_N, h_1, \dots, h_{2N+n+1}$ do not have common fixed points over $\partial\mathcal{S}$ since f_λ has no fixed points over $\Gamma_{f_\lambda}^{p_\lambda}$. Therefore, $f_1, \dots, f_N, h_1, \dots, h_{2N+n+1}$ do not have common fixed points in $\mathcal{S}$. Thus, there exists $\rho > 0$ satisfying the following condition:

$$d\big(x, h_\ell(x)\big) \geq \rho$$

for all $x \in \mathrm{Fix}(f_1, \dots, f_N, h_1, \dots, \widehat{h_\ell}, \dots, h_{2N+n+1}) \cap \mathcal{S}$ and for all $\ell \in \{1, \dots, 2N\}$.

Let $\delta > 0$ be such that the area of any disk of radius $\rho/3$ contained in Σ is greater than δ.

From Lemma 4.1 we have that the character curves

$$\Gamma_{h_1}^{q_1}, \dots, \Gamma_{h_N}^{q_N} \tag{5.1.1}$$

are contained in $\mathrm{Int}(\mathcal{S})$ and the distance between any two distinct curves of the above list is greater than or equal to ρ.

On the other hand it follows from the topology of $\mathcal{S}$ the existence of two distinct curves $\Gamma_{h_i}^{q_i}$ and $\Gamma_{h_j}^{q_j}$ in the list 5.1.1 which are homotopic. Furthermore, they cannot bound any disc in $\mathcal{S}$ since in that case, it would follow from Lemma 4.3 that $f_1, \ldots, f_N, h_1, \ldots, h_{2N+n+1}$ have common fixed points in the interior of the disc in question. This is however impossible.

Consequently, $\Gamma_{h_i}^{q_i}$ and $\Gamma_{h_j}^{q_j}$ bound a cylinder $\mathcal{C}_0 \subset \mathrm{Int}(\mathcal{S})$. In addition, $\mathcal{C}_0$ contains a 2-ball of radius $\rho/3$ since $d(\Gamma_{h_i}^{q_i}, \Gamma_{h_j}^{q_j}) \geq \rho$ and than the area $area(\mathcal{C}_0)$ of $\mathcal{C}_0$ is greater than δ.

Now, consider the compact surface $\mathcal{S} - \mathrm{Int}(\mathcal{C}_0)$ and let $\mathcal{S}_1 \subset \mathcal{S} - \mathrm{Int}(\mathcal{C}_0)$ be one of its connected components whose Euler characteristic is nonzero. We know that the connected components of $\partial\mathcal{S}_1$ are not null homotopic in Σ. Thus, $\partial\mathcal{S}_1$ has no more that N connected components. Now, let us choose N diffeomorphisms $\phi_1, \ldots, \phi_N$ in the list $h_1, \ldots, h_{2N}$ which are different from those used to construct the character curves in the boundary components of $\mathcal{S}_1$. Again, from the induction assumption and by using the above construction we obtain, for each $i \in \{1, \ldots, N\}$:

- a point $p(\phi_i) \in \mathrm{Int}(\mathcal{S}_1)$ which is a fixed point for all the diffeomorphisms $f_1, \ldots, f_N, h_1, \ldots, h_{2N+n+1}$ except for the diffeomorphism ϕ_i and such that $p(\phi_i)$ is a ω-recurrent point for ϕ_i;
- a character curve $\Gamma_{\phi_i}^{p(\phi_i)} \subset \mathrm{Int}(\mathcal{S}_1)$ of ϕ_i at $p(\phi_i)$.

Furthermore, since $\phi_1, \ldots, \phi_N$ are in the list $h_1, \ldots, h_{2N}$ it follows that the distance between any two distinct character curves of the list

$$\Gamma_{\phi_1}^{p(\phi_1)}, \ldots, \Gamma_{\phi_N}^{p(\phi_N)} \subset \mathrm{Int}(\mathcal{S}_1)$$

is greater than or equal to ρ. Once more, the topology of $\mathcal{S}_1$ implies that there exist two distinct curves $\Gamma_{\phi_i}^{p(\phi_i)}$ and $\Gamma_{\phi_j}^{p(\phi_j)}$ which are homotopic and do not bound disks in $\mathcal{S}_1$. Thus, they bound a cylinder $\mathcal{C}_1 \subset \mathrm{Int}(\mathcal{S}_1)$ such that $area(\mathcal{C}_1) > \delta$ since $d(\Gamma_{\phi_i}^{p(\phi_i)}, \Gamma_{\phi_j}^{p(\phi_j)}) \geq \rho$.

Consider now the compact surface $\mathcal{S}_1 - \mathrm{Int}(\mathcal{C}_1)$ and let $\mathcal{S}_2 \subset \mathcal{S}_1 - \mathrm{Int}(\mathcal{C}_1)$ be one of its connected components whose Euler characteristic is nonzero. Once again, we know that $\partial\mathcal{S}_2$ has no more than N connected components since they are not null homotopic in Σ. Now, consider N diffeomorphisms $\psi_1, \ldots, \psi_N$ in the list $h_1, \ldots, h_{2N}$ different from those used to construct the character curves in the boundary of $\mathcal{S}_2$. Repeating the construction above we obtain for each $i \in \{1, \ldots, N\}$:

- a point $p(\psi_i) \in \mathrm{Int}(\mathcal{S}_2)$ which is a fixed point for all the diffeomorphisms $f_1, \ldots, f_N, h_1, \ldots, h_{2N+n+1}$ except for the diffeomorphism ψ_i and such that $p(\psi_i)$ is a ω-recurrent point for ψ_i;
- a character curve $\Gamma_{\psi_i}^{p(\psi_i)}$ for ψ_i at $p(\psi_i)$ such that, two character curves satisfy $d\big(\Gamma_{\psi_i}^{p(\psi_i)}, \Gamma_{\psi_j}^{p(\psi_j)}\big) \geq \rho$ for all $i \neq j$ and $i, j \in \{1, \ldots, N\}$.

Applying this construction successively we obtain an infinite family of pairwise disjoint cylinders $(\mathcal{C}_i)_{i \geq 0}$ in $\mathrm{Int}(\mathcal{S})$ such that the area of each cylinder is greater than δ. This is a contradiction since the area of Σ is finite. □

Applying Theorem 5.1 for $\mathcal{S} = \Sigma$ we obtain:

Theorem 5.2. *If $f_1, \ldots, f_n \in \mathcal{V}_{3N+1}$ are commuting diffeomorphisms of Σ then they have common fixed points.*

Now, Theorem 1.1 follows from the above result and from the following reasoning which can be found in [8] and [3].

Let $\mathcal{F} \subset \mathcal{V}_{3N+1}$ be a nonempty set of commuting diffeomorphisms of Σ and let $\emptyset \neq \mathcal{G} \subset \mathcal{F}$ be a finite subset of $\mathcal{F}$. Then, by Theorem 5.2 it follows that $\mathrm{Fix}(\mathcal{G}) \neq \emptyset$. Hence, the family $\{\mathrm{Fix}(f)\}_{f\in\mathcal{F}}$ of closed subsets of Σ has the "finite intersection property" which implies that $\mathrm{Fix}(\mathcal{F}) \neq \emptyset$ and proves Theorem 1.1.

6. The main lemma

Now we prove the Main Lemma which is necessary to obtain Lemma 4.2 according to our strategy. We shall prove that the $k+1$ diffeomorphisms

$$f_0\,,\,\dots\,,\,f_{k-1}\,,\,f_k^{\tau_k}\circ\cdots\circ f_{3N}^{\tau_{3N}}$$

have a common fixed point in $\mathrm{Int}(\mathcal{S})$ provided that convenient character curves in the boundary of $\mathcal{S}$ and convenient exponents $\tau_k\,,\,\dots\,,\tau_{3N}$ are available.

For the sake of simplicity we use f_0 to denote the identity map of the surface Σ.

The proof of the Main Lemma will be by induction on k and by contradiction. We do it in two steps. The first one is Lemma 6.1 where we prove the case $k=1$, that is, we prove that $f_1^{\tau_1}\circ\cdots\circ f_{3N}^{\tau_{3N}}$ has a fixed point in $\mathrm{Int}(\mathcal{S})$ by means of the classical Poincaré Theorem for singularities of vector fields.

The second step is the Main Lemma itself where we prove the case $k>1$. It follows very closely the proof of Theorem 5.1.

In order to simplify the statement of the next lemmas we introduce the following definition.

Definition 6.1. Let $1 \le k \le 3N-1$ be an integer. Consider subsets $\Lambda_k\,,\dots,\Lambda_{3N} \subsetneq \{1,...,2N\}$ and integers $\tau_j \in \{1,\dots,2N\}-\Lambda_j$ for all $j \in \{k,\dots,3N\}$. Given a subset $\Lambda \subset \{1,...,3N\}$ together with commuting diffeomorphisms $f_1,\dots,f_{3N}$ one says that a simple closed curve $\Gamma \subset \Sigma$ is a *character curve associated to* $\Lambda_k,\dots,\Lambda_{3N}$, $\tau_k,\dots,\tau_{3N}$, Λ, $f_1,\dots,f_{3N}$ if:

(1) either there is $\lambda \in \Lambda$ and a point

$$p \in \mathrm{Fix}(f_1,\dots,\widehat{f_\lambda},\dots,f_{3N}) - \mathrm{Fix}(f_\lambda)$$

for which Γ is a character curve of f_λ at p;

(2) or there are $\xi \in \{k,\dots,3N-1\}$, a number $i \in \Lambda_\xi$ and a point

$$\mu \in \mathrm{Fix}(f_0,\dots,f_{\xi-1},f_\xi^i\circ f_{\xi+1}^{\tau_{\xi+1}}\circ\cdots\circ f_{3N}^{\tau_{3N}}) - \mathrm{Fix}(f_\xi)$$

such that Γ is a character curve of f_ξ at μ.

Now, we have the following lemma.

Lemma 6.1. *Consider subsets* $\Lambda_1,\dots,\Lambda_{3N} \subsetneq \{1,\dots,2N\}$*, integers* $\tau_j \in \{1,\dots,2N\}-\Lambda_j$ *for all* $j \in \{1,\dots,3N\}$ *and a subset* $\Lambda \subset \{1,\dots,3N\}$ *such that*

$$0 \le \#(\Lambda_1)+\cdots+\#(\Lambda_{3N})+\#(\Lambda) \le N.$$

Let $f_1,\dots,f_{3N} \in \mathcal{V}_{3N+1}$ *be commuting diffeomorphisms and let* $\mathcal{S} \subset \Sigma$ *be a compact connected surface with* $\chi(\mathcal{S}) \neq 0$*. Suppose that any connected component* Γ *of* $\partial\mathcal{S}$ *satisfies the following:*

- Γ *is not null homotopic in* Σ;
- Γ *is a character curve associated to* $\Lambda_1,\dots,\Lambda_{3N}$, $\tau_1,\dots,\tau_{3N}$, Λ, $f_1,\dots,f_{3N}$.

Then $f_1^{\tau_1}\circ f_2^{\tau_2}\circ\cdots\circ f_{3N}^{\tau_{3N}}$ *has a fixed point in* $\mathrm{Int}(\mathcal{S})$.

Proof. We have that $1\le\tau_1\le\cdots\le\tau_{3N}\le 2N$ and $f_1,\dots,f_{3N}\in\mathcal{V}_{3N+1}$. Then, it follows from the definition of the decreasing nested sequence $\{\mathcal{V}_k\}_{k\ge1}$ that $f_i\in\mathcal{V}_{3N+1}\subset\mathcal{V}_2$ and

$$f_1^{\tau_1}\circ\cdots\circ\widehat{f_i^{\tau_i}}\circ\cdots\circ f_{3N}^{\tau_{3N}}\in\mathcal{V}_{3N+1-(3N-1)}=\mathcal{V}_2\,.$$

Now, from the commutativity of $f_1,\dots,f_{3N}$ we conclude that

$$f_1^{\tau_1}\circ\cdots\circ f_{3N}^{\tau_{3N}}=\big(f_1^{\tau_1}\circ\cdots\circ\widehat{f_i^{\tau_i}}\circ\cdots\circ f_{3N}^{\tau_{3N}}\big)\circ f_i^{\tau_i}$$

has the form $h\circ f^{\ell}$ where $h\,,f\in\mathcal{V}_2\,,\ f\in\{f_1,\dots,f_{3N}\}$ and $1\le\ell\le 2N$.

Let us consider the first type of connected component of $\partial\mathcal{S}$, described in item (1) of Definition 6.1.

From item (2) of Lemma 3.1 we know that the character curve $\Gamma^p_{f_\lambda}$ is $\frac{\pi}{10}$-tangent to the vector field X_{f_λ}. Then, it follows from item (4) of Lemma 3.1 (case $i=0$ in item (4)) that the connected components of $\partial\mathcal{S}$ of type $\Gamma^p_{f_\lambda}$ are $\frac{2\pi}{5}$-tangent to the vector field $X_{f_1^{\tau_1}\circ\cdots\circ f_\lambda^{\tau_\lambda}\circ\cdots\circ f_{3N}^{\tau_{3N}}}$.

For the second type of connected component of $\partial\mathcal{S}$ we have the following. If $\Gamma^\mu_{f_\xi}$ is a connected component of $\partial\mathcal{S}$ for some integer $\xi\in\{1,\dots,3N-1\}$ then Definition 6.1 implies that

$$\mu\in\mathrm{Fix}(f_1^{\tau_1}\circ\cdots\circ f_\xi^{i}\circ f_{\xi+1}^{\tau_{\xi+1}}\circ\cdots\circ f_{3N}^{\tau_{3N}})-\mathrm{Fix}(f_\xi)$$

where $i\in\Lambda_\xi\subset\{1,\dots,2N\}$. Besides, item (2) of Lemma 3.1 yields that $\Gamma^\mu_{f_\xi}$ is $\frac{\pi}{10}$-tangent to the vector field X_{f_ξ}. Therefore, it follows from item (4) of Lemma 3.1 that $\Gamma^\mu_{f_\xi}$ is $\frac{2\pi}{5}$-tangent to the vector field $X_{f_1^{\tau_1}\circ\cdots\circ f_\xi^{\tau_\xi}\circ\cdots\circ f_{3N}^{\tau_{3N}}}$ since $\tau_\xi\in\{1,\dots,2N\}-\Lambda_\xi$.

Thus, $X_{f_1^{\tau_1}\circ\cdots\circ f_{3N}^{\tau_{3N}}}$ has a singularity in $\mathrm{Int}(\mathcal{S})$ by the classical Poincaré Theorem, since $\chi(\mathcal{S})\neq0$. Hence, the map $f_1^{\tau_1}\circ\cdots\circ f_{3N}^{\tau_{3N}}$ has a fixed point in $\mathrm{Int}(\mathcal{S})$ and the proof is finished. □

Main Lemma. *Consider subsets* $\Lambda_k,\dots,\Lambda_{3N}\subsetneq\{1,\dots,2N\}$, *integers* $\tau_j\in\{1,\dots,2N\}-\Lambda_j$ *for all* $j\in\{k,\dots,3N\}$ *and a subset* $\Lambda\subset\{1,\dots,3N\}$ *such that*

$$0\le\#(\Lambda_k)+\cdots+\#(\Lambda_{3N})+\#(\Lambda)\le N\quad\textit{where}\quad 1\le k\le 3N-1\,.$$

Let $f_1,\dots,f_{3N}\in\mathcal{V}_{3N+1}$ *be commuting diffeomorphisms and let* $\mathcal{S}\subset\Sigma$ *be a compact connected surface with* $\chi(\mathcal{S})\neq0$. *Suppose that any connected component* Γ *of* $\partial\mathcal{S}$ *satisfies the following*:

- Γ *is not null homotopic in* Σ;
- Γ *is a character curve associated to* $\Lambda_k,\dots,\Lambda_{3N}\,,\ \tau_k,\dots,\tau_{3N}\,,\ \Lambda\,,\ f_1,\dots,f_{3N}$.

Then the diffeomorphisms $f_0\,,\dots,f_{k-1}\,,f_k^{\tau_k}\circ\cdots\circ f_{3N}^{\tau_{3N}}$ *have common fixed points in* $\mathrm{Int}(\mathcal{S})$.

To prove, for example, that the case $k=1$ implies the case $k=2$ in the Main Lemma, we need to construct N convenient character curves supported by some special positive semi-orbits as in the proof of Theorem 5.1. For this we need to prove again that these semi-orbits stay in $\mathrm{Int}(\mathcal{S})$ and that the corresponding character curves are pairwise disjoint and do not intersect the character curves in the boundary of $\mathcal{S}$. These two fundamental steps are proved in the next two lemmas that we state without proof. Their proofs will be given in the last section. These two lemmas are actually a blend of Lemma 4.1 of [1, page 67] and Lemma 4.2 of [2, page 106] in a more general settings.

We recall that the notation $f_1,\dots,\widehat{f_\lambda},\dots,f_m$ means that f_λ is not in the list. Similarly, k_j is not in the list $k_1,\dots,\widehat{k_j},\dots,k_n$.

Lemma 6.2. *Let $f_1,\dots,f_{3N}\in\mathcal{V}_{3N+1}$ be commuting diffeomorphisms and let $1\le\lambda\,,\,\xi\le 3N$ be integers. Let $\Gamma_{f_\lambda}^{p_\lambda}$ and $\Gamma_{f_\xi}^{\mu_j}$ be supported by $\mathcal{O}^+_{p_\lambda}(f_\lambda)$ and $\mathcal{O}^+_{\mu_j}(f_\xi)$ respectively where*

- $p_\lambda\in\mathrm{Fix}(f_1,\dots,\widehat{f_\lambda},\dots,f_{3N})-\mathrm{Fix}(f_\lambda)\,;$
- $\mu_j\in\mathrm{Fix}(f_1^{k_1}\circ\cdots\circ f_\xi^{j}\circ\cdots\circ f_{3N}^{k_{3N}})-\mathrm{Fix}(f_\xi)\,;$

and $j,k_1,\dots,\widehat{k_\xi},\dots,k_{3N}\in\{1,\dots,2N\}$. Then, we have:

- $\gamma_{f_\xi}^{\mu_j}\cap\Gamma_{f_\lambda}^{p_\lambda}=\emptyset\,;$
- $\Gamma_{f_\xi}^{\mu_j}\cap\Gamma_{f_\lambda}^{p_\lambda}=\emptyset\,;$
- $\overline{\mathcal{O}^+_{\mu_j}(f_\xi)}\cap\Gamma_{f_\lambda}^{p_\lambda}=\emptyset.$

Moreover, let $\rho>0$ and $i\neq j$ with $i\,,j\in\{1,\dots,2N\}$ be such that

$$d\big(x\,,f_\xi(x)\big)\ge\rho\ ,\ \forall\,x\in\mathcal{O}^+_{\mu_i}(f_\xi)\cup\mathcal{O}^+_{\mu_j}(f_\xi).$$

Then $d(\Gamma_{f_\xi}^{\mu_i},\Gamma_{f_\xi}^{\mu_j})\ge\rho$.

Lemma 6.3. *Let $f_1,\dots,f_m\in\mathcal{V}_{3N+1}$ be commuting diffeomorphisms where $3\le m\le 3N$ and let $\Gamma_{f_1}^{\nu_j},\Gamma_{f_2}^{\mu_i}$ be curves supported by $\mathcal{O}^+_{\nu_j}(f_1)\,,\ \mathcal{O}^+_{\mu_i}(f_2)$ respectively satisfying:*

- $\nu_j\in\mathrm{Fix}(f_1^{j}\circ f_2^{k}\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m})-\mathrm{Fix}(f_1)$ *with* $1\le j\le 2N\,;$
- $\mu_i\in\mathrm{Fix}(f_1\,,f_2^{i}\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m})-\mathrm{Fix}(f_2)$ *with* $1\le i\le 2N$ *and* $i\neq k\,;$

where $k,k_3,\dots,k_m\in\{1,\dots,2N\}$. Then, we have:

- $\gamma_{f_1}^{\nu_j}\cap\gamma_{f_2}^{\mu_i}=\emptyset$ *and* $\Gamma_{f_1}^{\nu_j}\cap\Gamma_{f_2}^{\mu_i}=\emptyset\,;$
- $\gamma_{f_1}^{\nu_j}\cap\Gamma_{f_2}^{\mu_i}=\emptyset$ *and* $\Gamma_{f_1}^{\nu_j}\cap\gamma_{f_2}^{\mu_i}=\emptyset\,;$
- $\overline{\mathcal{O}^+_{\nu_j}(f_1)}\cap\Gamma_{f_2}^{\mu_i}=\emptyset$ *and* $\Gamma_{f_1}^{\nu_j}\cap\overline{\mathcal{O}^+_{\mu_i}(f_2)}=\emptyset.$

The next proof follows very closely the proof of Theorem 5.1.

6.1. **Proof of Main Lemma.** The proof will be by induction on $k\in\{1,\dots,3N-1\}$. The case $k=1$ is proved in Lemma 6.1.

Let us assume that the statement holds for some $k\in\{1,\dots,3N-2\}$. We will prove that it also holds for $k+1$.

For this, suppose for a contradiction that the diffeomorphisms

$$f_1\,,\,\dots\,,f_k\quad\text{and}\quad f_{k+1}^{\tau_{k+1}}\circ\cdots\circ f_{3N}^{\tau_{3N}}$$

have no common fixed points in $\mathrm{Int}(\mathcal{S})$. For each $j\in\{1,\dots,2N\}$ let us consider the maps

$$f_0\,,\,\dots\,,f_{k-1}\quad\text{and}\quad f_k^{j}\circ f_{k+1}^{\tau_{k+1}}\circ\cdots\circ f_{3N}^{\tau_{3N}}\,.$$

The induction assumption on k asserts that these maps have a common fixed point $\mu_{k,j}\in\mathrm{Int}(\mathcal{S})$ for all $j\in\{1,\dots,2N\}$ since we can take $\Lambda_k=\emptyset$ to apply the induction assumption. Moreover, we have:

- $f_k(\mu_{k,j})\neq\mu_{k,j}$ for all $j\in\{1,\dots,2N\}$ since the maps $f_1,\dots,f_k$ and $f_{k+1}^{\tau_{k+1}}\circ\cdots\circ f_{3N}^{\tau_{3N}}$ have no common fixed points in $\mathrm{Int}(\mathcal{S})\,;$
- $\overline{\mathcal{O}^+_{\mu_{k,j}}(f_k)}\subset\mathrm{Int}(\mathcal{S})$ which follows from Lemma 6.2, and from Lemma 6.3 since $\tau_\xi\in\{1,\dots,2N\}-\Lambda_\xi$ for all integer $\xi\in\{k+1,\dots,3N-1\}$. Lemma 6.2 guarantees us that $\gamma_{f_k}^{\mu_{k,j}}$ does not intersect the connected components $\Gamma_{f_\lambda}^{p}$ of $\partial\mathcal{S}$, described in item (1) of Definition 6.1. Lemma 6.3 assures that

$\gamma_{f_k}^{\mu_{k,j}}$ does not intersect the connected components $\Gamma_{f_\xi}^{\mu}$ of $\partial\mathcal{S}$, described in item (2) of Definition 6.1 for all $\xi \in \{k+1, \dots, 3N-1\}$.

By using Zorn's Lemma and the commutativity of the diffeomorphisms $f_1, \dots, f_{3N}$ one can suppose without loss of generality that $\mu_{k,j}$ is a ω-recurrent point for f_k. In that case, let $\Gamma_{f_k}^{\mu_{k,j}}$ be a character curve for f_k at $\mu_{k,j}$. By using Lemmas 6.2 and 6.3 as above we conclude that $\Gamma_{f_k}^{\mu_{k,j}} \subset \mathrm{Int}(\mathcal{S})$.

By assumption, the maps $f_1, \dots, f_k$ and $f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}$ have no common fixed points in $\mathrm{Int}(\mathcal{S})$. Moreover, it follows from item (3) of Lemma 3.1 that the map

$$f_1 \circ \cdots \circ f_k \circ f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}$$

do not have fixed points over $\partial\mathcal{S}$. Thus, $f_1, \dots, f_k$ and $f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}$ have no common fixed points over $\partial\mathcal{S}$. It results that $f_1, \dots, f_k$ and $f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}$ do not have common fixed points in $\mathcal{S}$. In such case, there exists $\rho > 0$ satisfying the following condition:

$$d\big(x, f_k(x)\big) \geq \rho$$

for all $x \in \mathrm{Fix}(f_0, \dots, f_{k-1}, f_k^{\ell} \circ f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}) \cap \mathcal{S}$ and for all integer ℓ such that $1 \leq \ell \leq 2N$.

Let $\delta > 0$ be such that the volume of any ball of radius $\rho/3$ contained in Σ is greater than δ.

Now, from Lemma 6.2 we have that the distance between any two distinct curves of the list of character curves

$$\Gamma_{f_k}^{\mu_{k,1}}, \dots, \Gamma_{f_k}^{\mu_{k,2N}} \subset \mathrm{Int}(\mathcal{S})$$

is greater than or equal to ρ. Moreover, the topology of $\mathcal{S} \subset \Sigma$ implies that in the list $\Gamma_{f_k}^{\mu_{k,1}}, \dots, \Gamma_{f_k}^{\mu_{k,N}}$ of N elements there exist two distinct curves $\Gamma_{f_k}^{\mu_{k,i}}$ and $\Gamma_{f_k}^{\mu_{k,j}}$ which are homotopic. On the other hand, it follows from Lemma 4.3 that these curves can not bound disks in $\mathcal{S}$. Thus, we conclude that these two curves bound a cylinder $\mathcal{C}_0 \subset \mathrm{Int}(\mathcal{S})$ which contains a ball of radius $\rho/3$ since $d(\Gamma_{f_k}^{\mu_{k,i}}, \Gamma_{f_k}^{\mu_{k,j}}) \geq \rho$. Consequently, the area $area(\mathcal{C}_0)$ of $\mathcal{C}_0$ is greater than δ.

Consider the compact surface $\mathcal{S} - \mathrm{Int}(\mathcal{C}_0)$ and let $\mathcal{S}_1 \subset \mathcal{S} - \mathrm{Int}(\mathcal{C}_0)$ be one of its connected components with nonzero Euler characteristic. We know that the connected components of $\partial\mathcal{S}_1$ are not null homotopic in Σ. Thus, $\mathcal{S}_1$ has no more then N connected components in its boundary.

At this step the connected components of $\partial\mathcal{S}_1$ are character curves associated to

$$\Lambda_k^1, \Lambda_{k+1}^1, \dots, \Lambda_{3N}^1, j, \tau_{k+1}, \dots, \tau_{3N}, \Lambda^1, f_1, \dots, f_{3N}$$

where:

- $\Lambda_k^1 = \big\{ i_k \ ; \ \Gamma_{f_k}^{\mu_{k,i_k}} \subset \partial\mathcal{S}_1 \big\} \subset \{1, \dots, N\}$;
- $\Lambda_\xi^1 \subset \Lambda_\xi$ for all $\xi \in \{k+1, \dots, 3N\}$ and $\Lambda^1 \subset \Lambda$;
- $j \in \{1, \dots, 2N\} - \Lambda_k^1$.

Of course, $\Lambda_k^1, \dots, \Lambda_{3N}^1, \Lambda^1$ are such that

$$\#(\Lambda_k^1) + \cdots + \#(\Lambda_{3N}^1) + \#(\Lambda^1) \leq N.$$

Now let us go back to the diffeomorphisms

$$f_0, \dots, f_{k-1} \quad \text{and} \quad f_k^j \circ f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}$$

and let us take $j \in \{1, \dots, 2N\} - \Lambda_k^1$. Note that $\#\big(\{1, \dots, 2N\} - \Lambda_k^1\big) \geq N$. From the induction assumption it results that they have common fixed points $\mu_{k,j} \in \mathrm{Int}(\mathcal{S}_1)$ for all $j \in \{1, \dots, 2N\} - \Lambda_k^1$. Repeating the same arguments as above we can assume that $\mu_{k,j}$ is ω-recurrent for f_k since $\overline{\mathcal{O}_{\mu_{k,j}}^+(f_k)} \subset \mathrm{Int}(\mathcal{S}_1)$. At this point we only need to verify that $\gamma_{f_k}^{\mu_{k,j}}$ do not intersect $\Gamma_{f_k}^{\mu_{k,i_k}}$ which is true since

$j \in \{1,\ldots,2N\} - \Lambda_k^1$ and $i_k \in \Lambda_k^1$. Following these arguments we conclude that there are two distinct curves $\Gamma_{f_k}^{\mu_{k,i}}$ and $\Gamma_{f_k}^{\mu_{k,j}}$ in the family of character curves contained in $\mathrm{Int}(\mathcal{S}_1)$

$$\left\{\Gamma_{f_k}^{\mu_{k,j}}\right\}_{j\in\{1,\ldots,2N\}-\Lambda_k^1}$$

which are not null homotopic in $\mathcal{S}_1$ and bound a cylinder $\mathcal{C}_1 \subset \mathrm{Int}(\mathcal{S}_1)$. Furthermore, $area(\mathcal{C}_1) > \delta$ since we have $d(\Gamma_{f_k}^{\mu_{k,i}}, \Gamma_{f_k}^{\mu_{k,j}}) \geq \rho$ for all $i\,, j \in \{1,\ldots,2N\}-\Lambda_k^1$ with $i \neq j$.

Consider now the compact surface $\mathcal{S}_1 - \mathrm{Int}(\mathcal{C}_1)$ and let $\mathcal{S}_2 \subset \mathcal{S}_1 - \mathrm{Int}(\mathcal{C}_1)$ be one of its connected components whose Euler characteristic is nonzero. Once again, we know that the connected components of $\partial\mathcal{S}_2$ are not null homotopic in Σ and, consequently, $\mathcal{S}_2$ has no more than N connected components in its boundary.

At this step the connected components of $\partial\mathcal{S}_2$ are character curves associated to

$$\Lambda_k^2\,, \Lambda_{k+1}^2\,, \ldots, \Lambda_{3N}^2\,, j, \tau_{k+1}, \ldots, \tau_{3N}, \Lambda^2, f_1\,, \ldots, f_{3N}$$

where:

- $\Lambda_k^2 = \left\{i_k\ ;\ \Gamma_{f_k}^{\mu_{k,i_k}} \subset \partial\mathcal{S}_2\right\} \subset \{1,\ldots,2N\}$;
- $\Lambda_\xi^2 \subset \Lambda_\xi^1$ for all $\xi \in \{k+1,\ldots,3N\}$ and $\Lambda^2 \subset \Lambda^1$;
- $j \in \{1,\ldots,2N\} - \Lambda_k^2$.

Of course, $\Lambda_k^2,\ldots,\Lambda_{3N}^2,\Lambda^2$ are such that

$$\#(\Lambda_k^2) + \cdots + \#(\Lambda_{3N}^2) + \#(\Lambda^2) \leq N.$$

Again, let us consider the diffeomorphisms

$$f_0\,,\ \ldots\,, f_{k-1} \quad \text{and} \quad f_k^j \circ f_{k+1}^{\tau_{k+1}} \circ \cdots \circ f_{3N}^{\tau_{3N}}$$

and let us take $j \in \{1,\ldots,2N\} - \Lambda_k^2$. Note that $\#\big(\{1,\ldots,2N\} - \Lambda_k^2\big) \geq N$. Once more, the induction assumption on k asserts that they have common fixed points $\mu_{k,j} \in \mathrm{Int}(\mathcal{S}_2)$ for all $j \in \{1,\ldots,2N\} - \Lambda_k^2$. Repeating exactly the same arguments as above we can assume that $\mu_{k,j}$ is ω-recurrent for f_k and we obtain two distinct curves $\Gamma_{f_k}^{\mu_{k,i}}, \Gamma_{f_k}^{\mu_{k,j}}$ in the family of character curves contained in $\mathrm{Int}(\mathcal{S}_2)$

$$\left\{\Gamma_{f_k}^{\mu_{k,j}}\right\}_{j\in\{1,\ldots,2N\}-\Lambda_k^2}$$

which are not null homotopic in $\mathcal{S}_2$ and bound a cylinder $\mathcal{C}_2 \subset \mathrm{Int}(\mathcal{S}_2)$. We have also that $area(\mathcal{C}_2) > \delta$ since the distance $d(\Gamma_{f_k}^{\mu_{k,i}}, \Gamma_{f_k}^{\mu_{k,j}}) \geq \rho$ for all $i\,, j \in \{1,\ldots,2N\} - \Lambda_k^2$ with $i \neq j$.

By successively repeating the above construction we obtain a family of pairwise disjoint cylinders $\{\mathcal{C}_i\}_{i\geq 0}$ contained in Σ and such that $area(\mathcal{C}_i) > \delta$ for all integer $i \geq 0$. This is however impossible and therefore completes the proof of the statement.

7. Proof of Lemma 4.2

Suppose for a contradiction that the diffeomorphisms $f_1,\ldots,f_{3N}$ have no common fixed points in $\mathrm{Int}(\mathcal{S})$. For each $j \in \{1,\ldots,2N\}$ let us consider the $3N-1$ diffeomorphisms

$$f_1\,,\ldots\,, f_{3N-2}\,, f_{3N-1}^j \circ f_{3N}.$$

Taking $k = 3N-1$ and $\Lambda_{3N-1} = \emptyset$ in the Main Lemma we conclude that they have a common fixed point $\mu_{3N-1,j} \in \mathrm{Int}(\mathcal{S})$ for all integer $j \in \{1,\ldots,2N\}$. Moreover, we have:

- $f_{3N-1}(\mu_{3N-1,j}) \neq \mu_{3N-1,j}$ for all $j \in \{1,\dots,2N\}$ since the diffeomorphisms $f_1,\dots,f_{3N}$ have no common fixed points in $\mathrm{Int}(\mathcal{S})$;
- $\overline{\mathcal{O}^+_{\mu_{3N-1,j}}(f_{3N-1})} \subset \mathrm{Int}(\mathcal{S})$ which follows from Lemma 6.2.

Zorn's Lemma and the commutativity of the maps $f_1,\dots,f_{3N}$ allow us to suppose without loss of generality that $\mu_{3N-1,j}$ is a ω-recurrent point for f_{3N-1}. In that case, let $\Gamma^{\mu_{3N-1,j}}_{f_{3N-1}}$ be a character curve for f_{3N-1} at $\mu_{3N-1,j}$. Once more, from Lemma 6.2 we have that $\Gamma^{\mu_{3N-1,j}}_{f_{3N-1}} \subset \mathrm{Int}(\mathcal{S})$.

From now on the proof of the lemma is concluded by repeating the proof of the Main Lemma: it suffices to substitute "k" by "$3N-1$" and the "induction assumption" by the "Main Lemma".

8. Proofs of technical lemmas

In this section we prove the last two technical lemmas. The proofs are similar but more technical than the proof of Lemma 4.1 because we treat the dynamics of curves supported by positive semi-orbits in a more general settings.

8.1. **Proof of Lemma 6.2.** Firstly note that for each $j \in \{1,\dots,2N\}$ the map $f_1^{k_1}\circ\cdots\circ f_\xi^{j}\circ\cdots\circ f_{3N}^{k_{3N}}$ has the form $h\circ f^\ell$ where the maps $h\,,f \in \mathcal{V}_2\,,\ f\in\{f_1,\dots,f_{3N}\}$ and the integer $\ell\in\{1,\dots,2N\}$.

Hence, it follows from item (3) of Lemma 3.1 (case $i=0$ in item (3)) that the diffeomorphism $f_1^{k_1}\circ\cdots\circ f_\xi^{j}\circ\cdots\circ f_{3N}^{k_{3N}}$ has no fixed points in the ball $\mathrm{B}\big(f_\lambda^m(p_\lambda)\,;3\,d(f_\lambda^m(p_\lambda)\,,f_\lambda^{m+1}(p_\lambda))\big)$.

Now, let us prove that $\Gamma^{\mu_j}_{f_\xi}\cap\Gamma^{p_\lambda}_{f_\lambda}=\emptyset$. For this, suppose for a contradiction that there exist $m,n,k,l\geq 0$ such that

$$[f_\lambda^m(p_\lambda)\,,f_\lambda^n(p_\lambda)]\cap[f_\xi^k(\mu_j)\,,f_\xi^l(\mu_j)]\neq\emptyset$$

where

$$\begin{aligned} d\big(f_\lambda^m(p_\lambda)\,,f_\lambda^n(p_\lambda)\big) &\leq \tfrac{3}{2}\,d\big(f_\lambda^m(p_\lambda)\,,f_\lambda^{m+1}(p_\lambda)\big)\\ d\big(f_\xi^k(\mu_j)\,,f_\xi^l(\mu_j)\big) &\leq \tfrac{3}{2}\,d\big(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j)\big). \end{aligned} \tag{8.0.1}$$

By the triangle inequality we have

$$\begin{aligned} d\big(f_\lambda^m(p_\lambda)\,,f_\xi^k(\mu_j)\big) &\leq d\big(f_\lambda^m(p_\lambda)\,,f_\lambda^n(p_\lambda)\big)+d\big(f_\xi^k(\mu_j)\,,f_\xi^l(\mu_j)\big)\\ &\leq \tfrac{3}{2}\,d\big(f_\lambda^m(p_\lambda)\,,f_\lambda^{m+1}(p_\lambda)\big)+\tfrac{3}{2}\,d\big(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j)\big)\\ &\leq 3\max\Big\{d\big(f_\lambda^m(p_\lambda)\,,f_\lambda^{m+1}(p_\lambda)\big)\,,\ d\big(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j)\big)\Big\}. \end{aligned}$$

Consequently, we have two possibilities:

- either $f_\xi^k(\mu_j)$ is in the ball $\mathrm{B}\big(f_\lambda^m(p_\lambda)\,;3\,d(f_\lambda^m(p_\lambda)\,,f_\lambda^{m+1}(p_\lambda))\big)$ which is however impossible as follows from the remarks below:
  - $f_\xi^k(\mu_j)\in\mathrm{Fix}(f_1^{k_1}\circ\cdots\circ f_\xi^{j}\circ\cdots\circ f_{3N}^{k_{3N}})$ by commutativity;
  - the diffeomorphism $f_1^{k_1}\circ\cdots\circ f_\xi^{j}\circ\cdots\circ f_{3N}^{k_{3N}}$ has no fixed points in the ball $\mathrm{B}\big(f_\lambda^m(p_\lambda)\,,3\,d\big(f_\lambda^m(p_\lambda)\,,f_\lambda^{m+1}(p_\lambda)\big)\big)$;
- or $f_\lambda^m(p_\lambda)$ is in the ball $\mathrm{B}\big(f_\xi^k(\mu_j)\,;3\,d(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j))\big)$:
  - If $\lambda\neq\xi$ then we have that $f_\lambda^m(p_\lambda)$ is a fixed point for f_ξ which is a contradiction since f_ξ does not have fixed points in the ball $\mathrm{B}\big(f_\xi^k(\mu_j)\,;3\,d(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j))\big)$ where $f_\xi^k(\mu_j)\notin\mathrm{Fix}(f_\xi)$;

- If $\lambda = \xi$ then we have $f_\lambda^m(p_\lambda) \in \mathrm{B}\big(f_\lambda^k(\mu_j)\,;3\,d(f_\lambda^k(\mu_j)\,,f_\lambda^{k+1}(\mu_j))\big)$ and $f_\lambda^m(p_\lambda) \in \mathrm{Fix}(f_1^{k_1}\circ\cdots\circ\widehat{f_\lambda^j}\circ\cdots\circ f_{3N}^{k_{3N}})$. On the other hand we have also that $f_\lambda^k(\mu_j) \in \mathrm{Fix}(f_1^{k_1}\circ\cdots\circ f_\lambda^j\circ\cdots\circ f_{3N}^{k_{3N}}) - \mathrm{Fix}(f_\lambda)$. Thus, it follows from item (3) of Lemma 3.1 that the diffeomorphism $f_1^{k_1}\circ\cdots\circ\widehat{f_\lambda^j}\circ\cdots\circ f_{3N}^{k_{3N}}$ has no fixed points in the ball $\mathrm{B}\big(f_\lambda^k(\mu_j)\,;3\,d(f_\lambda^k(\mu_j)\,,f_\lambda^{k+1}(\mu_j))\big)$ which is a contradiction.

The preceding discussion has shown that $\Gamma_{f_\xi}^{\mu_j}\cap\Gamma_{f_\lambda}^{p_\lambda} = \emptyset$. Similar arguments prove that $\gamma_{f_\xi}^{\mu_j}\cap\Gamma_{f_\lambda}^{p_\lambda} = \emptyset$ for all $j \in \{1,\ldots,2N\}$.

In addition, $\overline{\mathcal{O}_{\mu_j}^+(f_\xi)} \subset \mathrm{Fix}(f_1^{k_1}\circ\cdots\circ f_\xi^j\circ\cdots\circ f_{3N}^{k_{3N}})$ since the diffeomorphisms commute. Thus, it follows from item (3) of Lemma 3.1 that $\overline{\mathcal{O}_{\mu_j}^+(f_\xi)}\cap\Gamma_{f_\lambda}^{p_\lambda} = \emptyset$ for all $j\in\{1,\ldots,2N\}$ as we have seen in the beginning of the proof. This proves the first part of the lemma.

To prove the second part let $\rho>0$ and $i\neq j$ with $i\,,j\in\{1,\ldots,2N\}$ be such that

$$d(x\,,f_\xi(x)) \geq \rho \quad,\quad \forall\, x\in\mathcal{O}_{\mu_i}^+(f_\xi)\cup\mathcal{O}_{\mu_j}^+(f_\xi)\,.$$

Suppose that $d(\Gamma_{f_\xi}^{\mu_i},\Gamma_{f_\xi}^{\mu_j}) < \rho$. In that case there exist $m,n,k,l\geq 0$ satisfying (8.0.1) and two points $a\in[f_\xi^m(\mu_i)\,,f_\xi^n(\mu_i)]$ and $b\in[f_\xi^k(\mu_j)\,,f_\xi^l(\mu_j)]$ such that $d(a\,,b)<\rho$. Therefore,

$$\begin{aligned} d\big(f_\xi^m(\mu_i)\,,f_\xi^k(\mu_j)\big) &\leq d\big(f_\xi^m(\mu_i)\,,f_\xi^n(\mu_i)\big) + d(a\,,b) + d\big(f_\xi^k(\mu_j)\,,f_\xi^l(\mu_j)\big)\\ &\leq \tfrac{3}{2}\,d\big(f_\xi^m(\mu_i)\,,f_\xi^{m+1}(\mu_i)\big) + \rho + \tfrac{3}{2}\,d\big(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j)\big)\\ &\leq 4\max\Big\{d\big(f_\xi^m(\mu_i)\,,f_\xi^{m+1}(\mu_i)\big)\,,\,d\big(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j)\big)\Big\}.\end{aligned}$$

Consequently,

- either $f_\xi^k(\mu_j)$ is in the ball $\mathrm{B}\big(f_\xi^m(\mu_i)\,,4\,d\big(f_\xi^m(\mu_i)\,,f_\xi^{m+1}(\mu_i)\big)\big)$ which is impossible by item (3) of Lemma 3.1 since $f_\xi^k(\mu_j)$ is a fixed point of $f_1^{k_1}\circ\cdots\circ f_\xi^j\circ\cdots\circ f_{3N}^{k_{3N}}$ and $i\neq j$;
- or $f_\xi^m(\mu_i)$ is in the ball $\mathrm{B}\big(f_\xi^k(\mu_j)\,,4\,d\big(f_\xi^k(\mu_j)\,,f_\xi^{k+1}(\mu_j)\big)\big)$ which is impossible by the same reason.

The proof of Lemma 6.2 is finished.

8.2. **Proof of Lemma 6.3.** As remarked in the last proof, for each $j\in\{1,\ldots,2N\}$ the map $f_1^j\circ f_2^k\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m}$ has the form $h\circ f^\ell$ where $h\,,f\in\mathcal{V}_2\,,\ f\in\{f_1,\ldots,f_m\}$ and the integer ℓ lies in $\{1,\ldots,2N\}$.

Suppose that there exist $m,n,k,l\geq 0$ such that

$$[f_1^m(\nu_j)\,,f_1^n(\nu_j)]\cap[f_2^k(\mu_i)\,,f_2^l(\mu_i)]\neq\emptyset$$

where

$$\begin{aligned} d\big(f_1^m(\nu_j)\,,f_1^n(\nu_j)\big) &\leq \tfrac{3}{2}\,d\big(f_1^m(\nu_j)\,,f_1^{m+1}(\nu_j)\big)\\ d\big(f_2^k(\mu_i)\,,f_2^l(\mu_i)\big) &\leq \tfrac{3}{2}\,d\big(f_2^k(\mu_i)\,,f_2^{k+1}(\mu_i)\big).\end{aligned}$$

By the triangle inequality we have

$$\begin{aligned} d\big(f_1^m(\nu_j)\,,f_2^k(\mu_i)\big) &\leq d\big(f_1^m(\nu_j)\,,f_1^n(\nu_j)\big) + d\big(f_2^k(\mu_i)\,,f_2^l(\mu_i)\big)\\ &\leq \tfrac{3}{2}\,d\big(f_1^m(\nu_j)\,,f_1^{m+1}(\nu_j)\big) + \tfrac{3}{2}\,d\big(f_2^k(\mu_i)\,,f_2^{k+1}(\mu_i)\big)\\ &\leq 3\max\Big\{d\big(f_1^m(\nu_j)\,,f_1^{m+1}(\nu_j)\big)\,,\,d\big(f_2^k(\mu_i)\,,f_2^{k+1}(\mu_i)\big)\Big\}.\end{aligned}$$

Consequently,

- either $f_2^k(\mu_i)$ is in the ball $\mathrm{B}\big(f_1^m(\nu_j)\,;3\,d\big(f_1^m(\nu_j)\,,f_1^{m+1}(\nu_j)\big)\big)$ which is impossible since $f_2^k(\mu_i)\in\mathrm{Fix}(f_1)$ and f_1 has no fixed points in the ball $\mathrm{B}\big(f_1^m(\nu_j)\,;3\,d\big(f_1^m(\nu_j)\,,f_1^{m+1}(\nu_j)\big)\big)$ because $f_1^m(\nu_j)\notin\mathrm{Fix}(f_1)$;
- or $f_1^m(\nu_j)\in\mathrm{B}\big(f_2^k(\mu_i)\,;3\,d\big(f_2^k(\mu_i)\,,f_2^{k+1}(\mu_i)\big)\big)$ which is also impossible as follows from the following remarks:
  - $f_1^m(\nu_j)$ is a fixed point of $f_1^j\circ f_2^k\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m}$;
  - the diffeomorphism $f_1^j\circ f_2^k\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m}$ has no fixed points in the ball $\mathrm{B}\big(f_2^k(\mu_i)\,;3\,d\big(f_2^k(\mu_i)\,,f_2^{k+1}(\mu_i)\big)\big)$ when $i\neq k$ as a consequence of item (3) of Lemma 3.1. Note that we have $\mu_i\in\mathrm{Fix}(f_1^j\circ f_2^i\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m})$ and $\mu_i\notin\mathrm{Fix}(f_2)$.

This argument proves the first two items of the lemma.

To finish the proof we remark that $\overline{\mathcal{O}^+_{\nu_j}(f_1)}\subset\mathrm{Fix}(f_1^j\circ f_2^k\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m})$. On the other hand, $\mu_i\in\mathrm{Fix}(f_1^j\circ f_2^i\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m})-\mathrm{Fix}(f_2)$ and consequently, there is no fixed point of $f_1^j\circ f_2^k\circ f_3^{k_3}\circ\cdots\circ f_m^{k_m}$ over the curve $\Gamma_{f_2}^{\mu_i}$ since $i\neq k$. Thus, $\overline{\mathcal{O}^+_{\nu_j}(f_1)}\cap\Gamma_{f_2}^{\mu_i}=\emptyset$.

Finally, $\Gamma_{f_1}^{\nu_j}\cap\overline{\mathcal{O}^+_{\mu_i}(f_2)}=\emptyset$ since $\overline{\mathcal{O}^+_{\mu_i}(f_2)}\subset\mathrm{Fix}(f_1)$ and f_1 has no fixed points over $\Gamma_{f_1}^{\nu_j}$. The proof is finished.

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